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ELECTIVE COURSES IN MATHEMATICS FOR SECONDARY SCHOOLS.

A PRELIMINARY REPORT BY
THE NATIONAL COMMITTEE ON MATHEMATICAL REQUIREMENTS.

INTRODUCTION.

The Committee has elsewhere expressed its judgment that in the seventh, eighth and ninth grades mathematics should be a required subject. In the tenth, eleventh and twelfth grades, however, the extent to which election of subjects is permitted will depend on so many factors of a general character that it seems unnecessary and inexpedient for the present Committee to urge a positive requirement beyond the minimum one for the seventh, eighth and ninth grades. The subject must, like others, stand or fall on its intrinsic merit or on the estimate of such merit by the authorities responsible at a given time and place. The Committee believes, nevertheless, that every standard high school should not merely offer courses in mathematics for the tenth, eleventh and twelfth grades, but should encourage a large proportion of the pupils in its general courses to take some or all of these courses. Apart from the intrinsic interest and great educational value of the study of mathematics, it will in general be necessary for those preparing to enter college or to engage in the numerous occupations involving the use of mathematics to do work beyond the minimum requirement.

The present report is intended to suggest the most valuable mathematical training for students in general courses in secondary schools that will be supplementary to the first courses in mathematics as outlined in a previous report of the National

Committee.* Under present conditions most of this work will normally fall in the last two years of the high school, *i.e.*, in general, in the eleventh and twelfth school years.

The selection of material is based on the following general principles:†

1. The primary purpose of the teaching of mathematics should be to develop those powers of understanding and analyzing relations of quantity and of space which are necessary to a better appreciation of the progress of civilization and a better understanding of life and the universe about us and to develop those habits of thinking which will make these powers effective in the life of the individual.

2. The courses in each year should be so planned as to give the pupil the most valuable mathematical information and training which he is capable of receiving in that year with provision for his vocational and later educational needs.

The second principle leads us to the conclusion that the material for the elective courses offered should include, as far as possible, those mathematical ideas and processes that have the most important applications in the modern world. As a result we will naturally include certain material that at present is not ordinarily given in secondary courses; as, for instance, the material concerning the calculus. On the other hand we shall exclude certain other material that is now included in college entrance requirements. The results of an investigation made by the National Committee in connection with a study of these requirements indicates that modifications to meet these changes will be desirable from the standpoint of both college and secondary school.*

* Cf. Secondary School Circular No. 5 (February, 1920), U. S. Bureau of Education.

† The first principle is adopted unchanged from the report just referred to; the second principle is a modified form of the second principle of the same report. This modification is a logical consequence of the somewhat different aim of elective courses in mathematics as compared with courses for all students; for after students have completed the minimum essentials of mathematics it will be found desirable to differentiate their training according to special life aims, interests or aptitudes in so far as these are discoverable.

* Cf. "College Entrance Requirements in Mathematics," a preliminary report, to appear in the May, 1921, number of the MATHEMATICS TEACHER.

One cannot too strongly emphasize the fact that the broadening of content of high school courses in mathematics suggested in the report on the first courses and in the present report will very materially increase the usefulness of these courses to those who pursue them. It is of prime importance that educational administrators and others charged with the advising of students should take careful account of this fact in estimating the relative importance of mathematical courses and alternative elections. The number of important applications of mathematics in the activities of the world is today very large and is increasing at a rapid rate. This aspect of the progress of civilization has been noted by all observers who have combined a considerable knowledge of mathematics with an alert interest in the newer developments in other fields. It was revealed in very illuminating fashion during the recent war by the insistent demand for persons with varying degrees of mathematical training for many war activities of the first moment. Other forms of special training were also in demand, but in no single instance was the demand so widespread. If the same effort were made in time of peace to secure the highest level of efficiency available for the specific tasks of modern life, the demand for those trained in mathematics would be no less insistent. For it is in no wise true that the applications of mathematics in modern warfare are relatively more important or more numerous than its applications in those fields of human endeavor which are of a constructive nature.

There is another important point to be kept in mind in considering the relative value to the average student of mathematical courses and various alternative courses. If the student who omits the mathematical courses has need of them later, it is almost invariably more difficult and it is frequently impossible for him to obtain the training in which he is deficient. In the case of a considerable number of alternative subjects a proper amount of reading in spare hours at a more mature age will ordinarily furnish him the approximate equivalent to what he would have obtained in the way of information in a high school course in the same subject. It is not, however, possible to make up deficiencies in mathematical training in so simple a fashion. It requires systematic work under a com-

petent teacher to master properly the technique of the subject and any break in the continuity of the work is a handicap for which increased maturity rarely compensates. Moreover, when the individual discovers his need for further mathematical training, it is usually difficult for him to find the time from his other activities for systematic work in elementary mathematics.

Recommendations for Elective Courses.

The following topics are recommended for inclusion in the mathematical offerings to pupils who have satisfactorily completed the work outlined in the National Committee's report on The Reorganization of the First Courses in Secondary School Mathematics, comprising the elementary notions of Algebra, Intuitive Geometry, Numerical Trigonometry and Demonstrative Geometry.

1. **Algebra.** (*a*) *Simple functions of one variable.* Numerous illustrations and problems involving linear, quadratic and other simple functions including formulas from science and common life. More difficult problems in variation than those included in the earlier course.

(*b*) *Equations in one unknown.* Various methods for solving a quadratic equation (such as factoring, completing the square, use of formula) should be given. In connection with the treatment of the quadratic a very brief discussion of complex numbers should be included. Simple cases of the graphic solution of equations of degree higher than the second should be discussed and applied.

(*c*) *Equations in two or three unknowns.* The algebraic solution of linear equation in two or three unknowns and the graphic solution of linear equations in two unknowns should be given. The graphic and algebraic solution of a linear and a quadratic equation and of two quadratics that contain no first degree term and no xy term should be included.

(*d*) *Exponents, radicals and logarithms.* The definitions of negative, zero and fractional exponents should be given and it should be made clear that these definitions must be adopted if we wish such exponents to conform to the laws for positive integral exponents. Reduction of radical expressions to those

involving fractional exponents should be given as well as the inverse transformation. The rules for performing the fundamental operations on expressions involving radicals, and such transformations as

$$\sqrt[n]{a/b} = \frac{1}{b}\sqrt[n]{ab^{n-1}}, \quad \sqrt[n]{a^n b} = a\sqrt[n]{b}, \quad \frac{a}{\sqrt{b}+\sqrt{c}} = \frac{a(\sqrt{b}-\sqrt{c})}{b-c}$$

should be included. In close connection with the work on exponents and radicals there should be given as much of the theory of logarithms as is involved in their application to computation and sufficient practice in their use in computation to impart a fair degree of facility.

(*e*) *Arithmetic and Geometric Progressions.* The formulas for the nth term and the sum of n terms should be derived and applied to significant problems.

(*f*) *Binominal Theorem.* A proof for positive integral exponents should be given; it may also be stated that the formula applies to the case of negative and fractional exponents under suitable restrictions, and the problems may include the use of the formula in these cases as well as in the case of positive integral exponents.

2. **Solid Geometry.** The aim of the work in solid geometry should be to exercise further the spatial imagination of the student and to give him a knowledge of the fundamental spatial relationships and power to work with them. For many of the practical applications of mathematics it is of fundamental importance to have accurate space perceptions. Hence it would seem wise to have at least some of the work in solid geometry come as early as possible in the mathematical courses, preferably not later than the beginning of the third year of high school. For schools that can complete more than the preliminary courses outlined in the previous report during the first two years it would seem that the more elementary notions of solid geometry might well be listed among the optional topics of the first two years to be studied in connection with related ideas of plane geometry.

The work in solid geometry should include numerous exercises in computation based on the formulas established. This will serve to correlate the work with arithmetic and algebra

and to furnish practice in computation. It is of the first importance that such practice be continued throughout the entire mathematical course. For a minimum course it will be possible to omit a considerable number of propositions ordinarily given and many of the exercises of a more theoretical nature. For example, proofs of the more difficult propositions dealing with volumes and areas, that can be established more readily by the methods of the calculus, may well be postponed until they can be discussed as applications of the latter subject. The minimum course should certainly include propositions dealing with perpendiculars to planes, dihedral angles, and the simpler theorems on areas and volumes. It should be possible to complete such a minimum course in a third of a year.*

3. **Trigonometry**. The work in elementary trigonometry begun in the earlier years should be completed by including the logarithmic solution of right and oblique triangles, radian measure, graphs of trigonometric functions, the derivation of the fundamental relations between the functions and their use in providing identities and in solving easy trigonometric equations. There should be, wherever practicable, some use of the transit in connection with the simpler operations of surveying and of the sextant for some of the simpler astronomical observations, such as those involved in finding local time. When no transit or sextant is available, simple apparatus for measuring angles roughly may and should be improvised. Drawings to scale should form an essential part of the numerical work in trigonometry. The use of the slide rule in computations requiring only three place accuracy and in checking other computations is also recommended.

4. **Elementary statistics.** Continuation of the earlier work to include the meaning and use of fundamental concepts and simple frequency distributions with graphic representations of various kinds and measures of central tendency.

* Some European schools have found it desirable to replace some of the work now usually given in solid geometry by certain important topics of descriptive geometry. Since no textbook is at present available for this purpose the Committee refrains from any recommendation in this direction. The possibility of a scientific and logical treatment of descriptive geometry would seem to be worthy of the attention of teachers, however.

5. **Elementary calculus.** The work should include:

(*a*) The general notion of a derivative as a limit indispensable for the accurate expression of such fundamental quantities as velocity of a moving body or slope of a curve.

(*b*) Applications of derivatives to easy problems in rates and in maxima and minima.

(*c*) Simple cases of inverse problems; e.g., finding distance from velocity, etc.

(*d*) Approximate methods of summation leading up to integration as a powerful method of summation.

(*e*) Applications to simple cases of motion, area, volume and pressure.

The work in calculus should be largely graphic and closely related to that in physics; the necessary technique should be reduced to a minimum by basing it wholly or mainly on algebraic polynomials. No formal study of analytic geometry need be presupposed beyond the plotting of simple graphs.

It is important to bear in mind that while the elementary calculus is sufficiently easy and interesting to justify its introduction special pains should be taken to guard against any lack of thoroughness in the fundamentals of algebra and geometry. No possible gain could compensate for a real sacrifice of such thoroughness.

It should also be borne in mind that the suggestion of including elementary calculus is not intended for all schools nor for all teachers or all pupils in any school. It is not intended to connect in any direct way with college entrance requirements. The future college student will have ample opportunity for calculus later. The capable boy or girl who is not to have the college work ought not on that account to be prevented from learning something of the use of this powerful tool. The applications of elementary calculus to simple concrete problems are far more abundant and more interesting than those of algebra. The necessary technique is extremely simple. The subject is commonly taught in secondary schools in England, France and Germany and appropriate English texts are available.*

6. **History and Biography.** Historical and biographical

* Quotations and typical problems from one of these texts will be found in a supplementary note appended to this report.

material should be used throughout to make the work more interesting and significant.

At the present time these topics (1–5) can probably in most high schools be given most advantageously as separate units of a two-year program. However, the National Committee is of the opinion that methods of organization are being experimentally perfected whereby teachers will be enabled to present the same material more effectively in combined courses unified by one or more central ideas.

Additional Electives.

Such additional electives as *mathematics of investment*, *shop mathematics*, *surveying and navigation*, *descriptive or projective geometry* will appropriately be offered by schools which have special needs or conditions, but it seems unwise for the National Committee to attempt to define them pending the results of further experience on the part of these schools.

Supplementary Note on the Calculus as a High-School Subject.

In connection with the recommendations concerning the calculus, such questions as the following may arise: Why should a college subject like this be added to a high school program? How can it be expected that high-school teachers will have the necessary training and attainments for teaching it? Will not the attempt to teach such a subject result in loss of thoroughness in earlier work? Will anything be gained beyond a mere smattering of the theory? Will the boy or girl ever use the information or training secured? The subsequent remarks are intended to answer such objections as these and to develop more fully the point of view of the Committee in recommending the inclusion of elementary work in the calculus in the high school program.

By the calculus we mean for the present purpose a study of *rates of change*. In nature all things change. How much do they change in a given time? How fast do they change? Do they increase or decrease? When does a changing quantity become largest or smallest? How can rates of changing quantities be compared?

These are some of the questions which lead us to study the elementary calculus. Without its essential principles these questions cannot be answered with definiteness.

The following are a few of the specific replies that might be given in answer to the questions listed at the beginning of this note: The difficulties of the college calculus lie mainly outside the boundaries of the proposed work. The elements of the subject present less difficulty than many topics now offered in advanced algebra. It is not implied that in the near future

many secondary-school teachers will have any occasion to teach the elementary calculus. It is the culminating subject in a series which only relatively strong schools will complete and only then for a selected group of students. In such schools there should always be teachers competent to teach the elementary calculus here intended. No superficial study of calculus should be regarded as justifying any substantial sacrifice of thoroughness. In the judgment of the Committee the introduction of elementary calculus necessarily includes sufficient algebra and geometry to compensate for whatever diversion of time from these subjects would be implied.

The calculus of the algebraic polynominal is so simple that a boy or girl who is capable of grasping the idea of limit, of slope, and of velocity, may in a brief time gain an outlook upon the field of mechanics and other exact sciences, and acquire a fair degree of facility in using one of the most powerful tools of mathematics, together with the capacity for solving a number of interesting problems. Moreover, the fundamental ideas involved, quite aside from their technical applications, will provide valuable training in understanding and analyzing quantitative relations,—and such training is of value to everyone.

The following typical extracts from an English text intended for use in secondary schools may be quoted:

"It has been said that the calculus is that branch of mathematics which schoolboys understand and senior wranglers fail to comprehend. . . . So long as the graphic treatment and practical applications of the calculus are kept in view, the subject is an extremely easy and attractive one. Boys can be taught the subject early in their mathematical career, and there is no part of their mathematical training that they enjoy better or which opens up to them wider fields of useful exploration. . . . The phenomena must first be known practically and then studied philosophically. To reverse the order of these processes is impossible."

The text in question, after an interesting historical sketch, deals with such problems as the following:

A train is going at the rate of 40 miles an hour. Represent this graphically.

At what rate is the length of the daylight increasing or decreasing on December 31, March 26, etc.? (From tabular data.)

A cart going at the rate of 5 miles per hour passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles an hour, passes the same milestone. Find when and where the bicycle will overtake the cart.

A man has four miles of fencing wire and wishes to fence in a rectangular piece of prairie land through which a straight river flows, the bank of the stream being utilized as one side of the enclosure. How can he do this so as to enclose as much land as possible?

A circular tin canister closed at both ends has a surface area of 100 sq. cm. Find the greatest volume it can contain.

Post office regulations prescribe that the combined length and girth of a parcel must not exceed 6 feet. Find the maximum volume of a parcel whose shape is a prism with the ends square.

A pulley is fixed 15 feet above the ground, over which passes a rope 30 feet long with one end attached to a weight which can hang freely, and the other end is held by a man at a height of 3 feet from the ground. The man walks horizontally away from beneath the pulley at the rate of 3 feet per second. Find the rate at which the weight rises when it is 10 feet above the ground.

The pressure on the surface of a lake due to the atmosphere is known to be 14 lb per sq. in. The pressure in the liquid x inches below the surface is known to be given by the law $dp/dx = 0.036$. Find the pressure in the liquid at a depth of 10 feet.

The arch of a bridge is parabolic in form. It is 5 feet wide at the base and 5 feet high. Find the volume of water that passes through per second in a flood when the water is rushing at the rate of 10 feet per second.

A force of 20 tons compresses the spring buffer of a railway stop through 1 inch, and the force is always proportional to the compression produced. Find the work done by a train which compresses a pair of such stops through six inches.

These may illustrate the aims and point of view of the proposed work. It will be noted that not all of them involve calculus, but those that do not lead up to it.

Comments, criticisms and suggestions for the revision of this preliminary report are urgently requested. They should be sent as soon as possible to J. W. Young, chairman, Hanover. N. H.

ALIGNMENT CHARTS.

The subject of nomographic or alignment charts has received considerable attention during the last few years. Although its value as a time and labor saving device is so evident, still it has taken almost a generation since its inception by the French engineer and mathematician, M. D'Ocagne, before the American engineer has sought to profit by it. The world war brought our ordnance engineers in contact with the French engineers, and the former have learned how the latter apply the principles underlying the alignment chart to the graphical solution of some of their problems in ballistics and allied subjects. To-day, some of our manufacturers are becoming interested in these charts, and the "Department of Industrial Cooperation and Research" at the Massachusetts Institute of Technology, which is in close contact with over two hundred of these firms, has received many requests for alignment chart solutions of various simple problems which have arisen in their shop work. These solutions, because of their simplicity, can be used by the workmen in the shop with considerable facility and little chance of error.

The principles upon which the alignment chart is based are of the simplest—the idea of the representation of a function of a variable by a scale, and the well-known theorem that in similar triangles corresponding sides are proportional—ideas which the average secondary school student easily grasps. A few simple examples will suffice to illustrate some of the methods employed.*

The student is familiar with various scales—the straight thermometer scale with its uniform divisions and the uniform

* For a complete exposition of all the methods, the reader may be referred to the admirable treatise by M. D'Ocagne, "Traité de Nomographie," Paris, Gauthier-Villars. In the English language reference may be made to S. Brodetsky, "A First Course in Nomography," London, G. Bell & Sons; J. B. Peddle, "The Construction of Graphical Charts," New York, McGraw-Hill; J. Lipka, "Graphical and Mechanical Computation," Part I, "Alignment Charts," New York, John Wiley & Sons.

circular scale of the protractor, the non-uniform logarithmic scale of the slide rule, the non-uniform curved scale of a galvanometer dial, etc.—and he also knows that a change in the unit of measurement used in laying out such a scale will merely change its length without changing its nature or properties. Thus, in ex. 1, $OA = \frac{3}{10}\, p$ inches, $\frac{3}{10}''$ being the unit used in laying off the scale for the variable p,* while $OB = \frac{1}{4}\, v$ inches $\frac{1}{4}''$ being the unit used in laying off the scale for the variable v. Again, in ex. 4, $OA = 10 \log x$ inches, $10''$ being the unit used in laying off the scale for the variable x, so that the stroke marked $x = 2$ is at a distance $10 \log 2 = 3.01''$ from O.

Ex. 1. A manufacturer asked for a graphical solution of the following problem: *Out of a piece of cloth 36″ long and of width varying from 15″ to 40″, rectangular pieces of cloth were to be cut varying in dimensions from 3″ × 4″ to 10″ × 20″. To find the percentage any such rectangular piece is of the whole piece.*

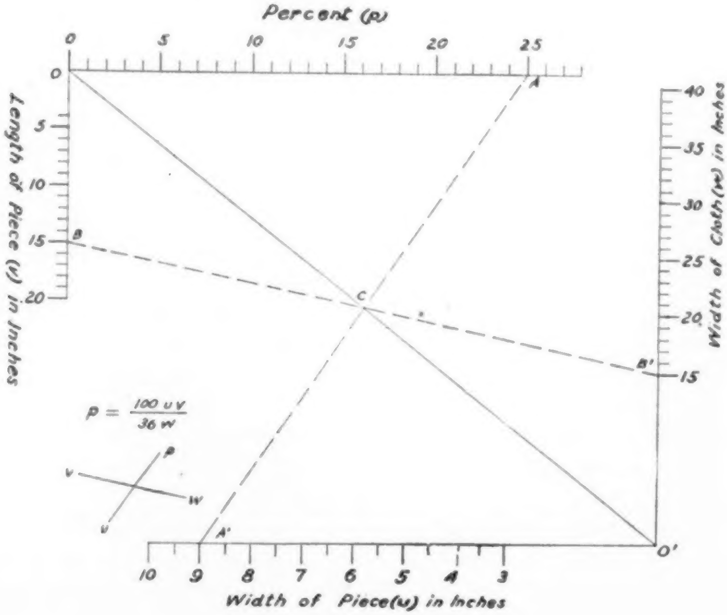

FIG. 1.

* The units are used in the construction of the original charts; the figures in the text are reductions of the original drawings.

If we let w represent the varying width of the whole piece of cloth, u and v the varying dimensions of the rectangular piece, and p the required percentage, we have

$$p = \frac{100uv}{36w}, \tag{1}$$

or

$$p : \tfrac{25}{9}u = v : w. \tag{2}$$

In Fig. 1, we have drawn from any two convenient points O and O', the parallel lines OA and $O'A'$ and the parallel lines OB and $O'B'$. If the dotted lines AA' and BB' are any two lines cutting the oblique line OO' in the same point C, then by similar triangles we have

$$OA : O'A' = OB : O'B'. \tag{3}$$

Now, on the two sets of parallel lines we construct scales for the variables, p, u, v, and w with m_1, m_2, m_3, and m_4 inches, respectively, as units of measurements, *i.e.*, so that

$$OA = m_1 p,\ O'A' = m_2 u,\ OB = m_3 v,\ O'B' = m_4 w.$$

Substituting these values in (3), we note that in order that this equation should reduce to (2), we must have

$$m_1 : \tfrac{9}{25} m_2 = m_3 : m_4. \tag{4}$$

This is the only relation to be satisfied by our units. We choose our units so that the scales should have convenient lengths. In constructing Fig. 1, we used

$$m_1 = \tfrac{3}{10}'',\ m_2 = \tfrac{5}{6}'',\ m_3 = \tfrac{1}{4}'',\ m_4 = \tfrac{1}{4}''. \tag{5}$$

The scales may be laid off with a properly graduated straight edge.

The chart is now ready for use. Any two dotted lines (called index lines), cutting the oblique line OO' in the same point, will cut the scales in values of p, u, v, and w satisfying the given equation. Given any three of the variables the fourth may be found. In practice the index lines are not actually drawn. The figure is drawn in ink on heavy bristol board; a blue print may be made. If, for example, u, v and w are given,

we lay a straight edge through the proper values of u and v, marking its point of intersection C with OO', then rotate the straight edge about C until it passes through the proper value of w, and then read the required value of p from its intersection with the p-scale.

Ex. 2. A graphical solution of the following problem was called for. *Given two numbers, to find the percentage which one of the numbers is of their sum.*

If the numbers are x and y, and z is the percentage which x is of $x + y$, we have

$$(6) \qquad z = \frac{100x}{x + y}, \quad \text{or} \quad x + y = \frac{x}{.01z}.$$

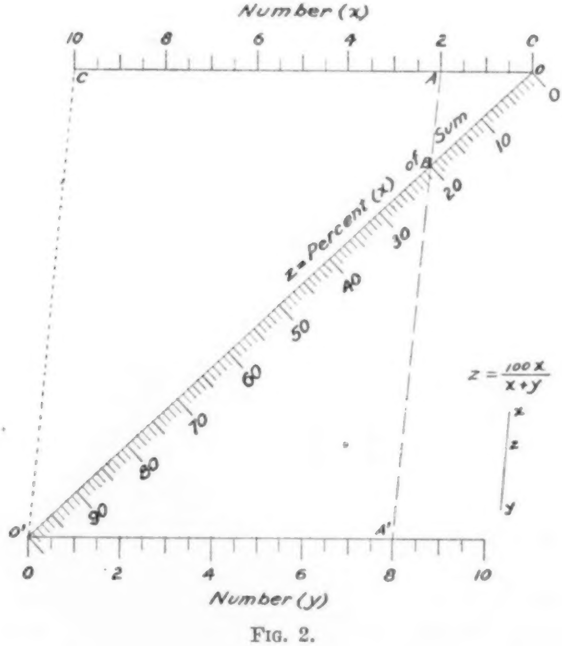

FIG. 2.

In Fig. 2, we have drawn the parallel lines OA and $O'A'$ from any two convenient points O and O', and joined OO'. If the dotted line is any transversal cutting the parallels in A and A' and the oblique line OO' in B, and we draw the construction line $O'C$ parallel to AA', then by similar triangles, we have

(7) $OC:OO' = OA:OB$, or $OA + O'A' : OO' = OA:OB$.
Now, if on the parallel lines we construct scales for x and y with the same unit m_1 inches, so that

$$OA = m_1x,\ O'A' = m_1y,$$

and on OO', starting at O, we construct a scale for z with a unit m_2 inches, so that $OB = m_2z$, then (7) becomes

$$m_1(x+y) : OO' = m_1x : m_2z,$$

and in order that this should reduce to (6), we must have

(8) $$m_1 : OO' = m_1 : 100m_2$$

This is the only relation to be satisfied by the units. For the con-

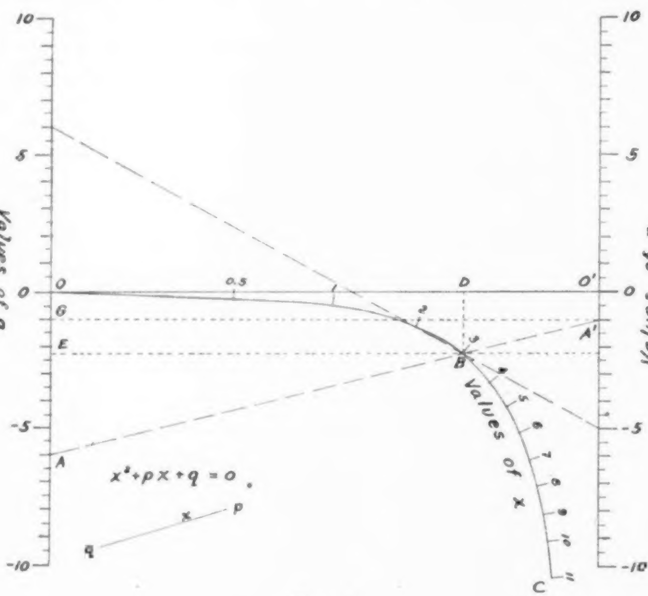

FIG. 3.

struction of Fig. 2, we choose $OO' = 10''$, $m_1 = 1''$, $m_2 = 0.1''$, so that

(9) $$OA = x,\ O'A' = y,\ OB = 0.1z.$$

Any index lines will cut the three scales in values satisfying

equation (6). We have constructed the chart for values of x and y between 0 and 10.

Ex. 3. To find graphically the roots of any quadratic equation. If we consider the coefficients p and q as variables, the equation

$$x^2 + px + q = 0 \tag{10}$$

may be said to represent all quadratic equations.

In Fig. 3, the directions of the lines are important—the convention being the same as in graphical algebra. We have drawn two parallel lines OA and $O'A'$ any convenient distance apart, a curve OBC, and any transversal AA' cutting the parallels in A and A' and the curve in B. The line OO', perpendicular to the parallels, is 20″ long. The lines OA and $O'A'$ carry scales for q and p respectively with units of 1″, so that $OA = q$, $O'A' = p$, and p and q vary from -10 to $+10$. The curve OBC and its accompanying scale are constructed by assigning a value to x, and laying off

$$OD = \frac{20x}{x+1}, \quad DB = -\frac{x^2}{x+1} \tag{11}$$

and marking the point reached with this value of x. This process is continued for values of x from 0 to 11 for as small intervals as we please, the points being marked with their proper values of x, and a smooth curve is drawn through the points obtained.

If we draw the construction lines BE and $A'G$ parallel to $O'O$, then by similar triangles,

$$EA : GA = EB : GA',$$

or

$$OA - DB : OA - O'A' = OD : OO',$$

or

$$q + \frac{x^2}{x+1} : q - p = \frac{20x}{x+1} : 20,$$

or

$$x^2 + px + q = 0,$$

so that any index line will cut the scales in values of p, q, and x satisfying this equation, or any index line cutting the parallel

scales in p and q will cut the curved scale in values of x which are the roots of the quadratic equation.

We have only constructed the curve for positive values of x. If the roots are negative, we merely change the sign of p, and proceed to find the positive roots of the new equation; these will be the negative roots of the given equation. If the index line does not cut the curve the roots are imaginary, and if it is tangent to the curve the roots are equal. If the values of p and q lie beyond the limits -10 and $+10$, it is merely necessary to make the transformation $x = ax'$, and get

$$x'^2 + \frac{p}{a}x' + \frac{q}{a^2} = 0,$$

and then choose a so that the new coefficients

$$p' = \frac{p}{a}$$

and

$$q' = \frac{q}{a^2}$$

lie between -10 and $+10$.

The solution of any cubic equation $x^3 + px + q = 0$ may be solved graphically in a similar manner. The corresponding curved scale is obtained by laying off

$$OD = \frac{20x}{x+1}, \quad DB = -\frac{x^3}{x+1}.$$

Ex. 4. To build a graphical chart for multiplication and division. Let $z = xy$. This may be written

$$\log z = \log x + \log y. \tag{12}$$

In Fig. 4, we have drawn three parallel equidistant lines and any transversal cutting these in A, A', and A''. Then evidently

$$O''A'' = \tfrac{1}{2}(OA + O'A').$$

If we construct scales on these lines so that

$$OA = 10 \log x, \quad O'A' = 10 \log y, \quad O''A'' = 5 \log z,$$

where we have used 10″, 10″, and 5″ as our units, and mark the scales with corresponding values of x, y, and z, then any

straight edge laid across the scales will cut out values of x, y, and z such that $\log z = \log x + \log y$ or $z = xy$. Multiplication and division are performed by the same process. The logarithmic scales are approximate reproductions of the scales

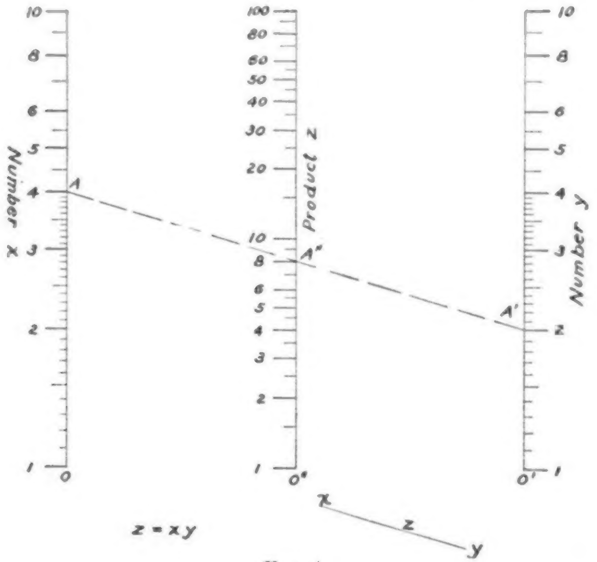

FIG. 4.

of the slide rule—the scales of the slide rule have 25 cm. and 12.5 cm. as units. Of course, as in the slide rule, the decimal point may be disregarded and its final position determined from the problem.

The method outlined in this example has a wider application than any of the other methods, for it may easily be extended to any equation in three variables of the form $z^n = ax^m y^k$, where a, k, m, and n are any constants whatsoever; these constants merely determine the relative positions of the scales and the units of measurements to be used in the construction of the scales. The method may also be extended to formulas involving more than three variables.

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A REORGANIZED COURSE IN JUNIOR HIGH-SCHOOL ARITHMETIC

The National Committee on Mathematical Requirements, in its preliminary report on Junior High School Mathematics, invites "criticisms and discussions" of the report and offers to serve as a "clearing-house of ideas and material" sent to the committee.

The material here submitted has been tried out in the seventh and eighth grade classes of Fairmount Junior High School in Cleveland, Ohio, with very satisfactory results. Its place in the curriculum, however, is not of as much concern as is the unbroken chain of related topics. The plan is to present, as one unit, the cumulative business ideas which come under the heading, "The Applications of Percentage."

The pupils are supposed to have been well drilled in the four fundamentals with respect to integers, common and decimal fractions, and the interchanging of common fractions and percents. The meaning of per cent., finding any per cent. of a number, finding what per cent. one number is of another, finding a number when a certain per cent. of it is known, become more familiar when used in connection with the work to be outlined, though there should be some previous understanding of these operations. The equation method of solving indirect problems is gradually developed by using statements which, when abbreviated, form an equation. The pupils readily see the reason for, and the meaning of the abbreviations and the simpler equation principles are accepted and used. The use of this method is never forced, but is considered a convenience.

The necessity of presenting to the pupils' minds new material for thought which can be attached to ideas already known by them, is recognized, and this recognition has been a guide in the formation of the following outline.

The activity of buying and selling things for the purpose of making money is very familiar to pupils and is therefore the first topic studied. There is invariably, already in the pupils'

minds, a wealth of ideas on this subject commonly named Profit and Loss, and the accumulation of information begins by the teacher suggesting business transactions which will naturally introduce the ideas of gross gain, overhead expenses, net gain, loss, gain and loss per cents.

When these ideas are firmly fixed in mind by much oral and written work, questions concerning the growth of a business are brought up. One of the overhead expenses mentioned was, no doubt, advertising. Having in mind the topic Commercial Discount, the teacher asks the pupils to bring to class newspaper advertisements, announcing sales, which have the former price as well as the sale price given. In this way commercial discount is seen to be a very common and popular business activity. In making comparisons of the discounts allowed on articles in one or several advertisements, the need of the basic per cent. standard is felt and rates of discounts are figured and studied with intelligence and interest.

Discussions about the sales, as to whether they aid or hinder business, keep in mind the previous work. In these discussions the pupils give probable reasons for the discount sales. Other reasons for discounts such as cash payments, being an employee in a store, etc., should be brought to mind.

The subject of *successive discounts*, generally such a stumbling block, has been presented by imagining a clerk in a store which allows its employees a 10 per cent. discount on articles purchased by them from the store. The clerk buys an article which is selling at a 25 per cent. discount. If allowed his 10 per cent. discount on sales, as is often the case, the sum discounted would be the sale price, not the marked price. Such an example of successive discounts is followed by other cases, such as when several reductions are allowed on catalogued prices.

Do salesmen increase the business of a firm? Comparisons of salesmen having characteristics which are detrimental or advantageous to their selling ability, are made. Would it be fair to pay them all the same salary? If not, what is a fair way of remunerating them? By such questions *commission*, the method used in paying newsboys as well as high class salesmen for services rendered, is introduced. The various kinds of

work paid for on the commission basis are introduced in problems. The reasonableness of the accepted rule that the commission agent receives a certain per cent. of the money received or spent for goods or collected for his employer, is recognized.

By this time the problems begin to show the cumulated ideas. Bills, properly made out, totaled, discounted, and receipted, may be used to figure the salesman's commission. Commission is considered an aid to business and that is kept in the minds of the pupils, by the teacher, in guiding discussions and giving examples.

We cannot go very far in the study of business transactions without understanding situations involving the borrowing and loaning of money. We reach the subject of *interest* on money through the channel of *rent*. Articles may be bought and sold for profit. They may also be loaned for profit. In considering various things which may be rented, the time element comes in. A house may be rented by the month, an automobile by the hour, a summer cottage for a six-month period, etc. When money is rented, a certain per cent. of it is charged for a year's time. So we obtain the rule for finding the interest on money loaned:

> The principal, multiplied by the rate per cent., multiplied by the time expressed in years, equals the interest.

This rule shortened into the formula, $P \times R \times T = I$, affords a means of solving interest problems requiring the time, or the rate, or the principal, as well as the interest.

The transition to the study of *banking* follows naturally. Three important functions are considered: (1) That of the savings department, (2) that of the checking department, and (3) that of the loaning department.

The first function presents the twofold idea of an accommodation to the depositor as well as to the bank. The depositor loans the money to the bank, the bank pays interest for the money. Much use is made of bank forms easily obtained from banks: The identification card, bank book, deposit slip, withdrawal slip, etc. By making in the bank book entries of interest when due, the idea of *compound interest* is clarified,

and an incentive to save is born in the mind of many a pupil. When the principal of compound interest is understood, use is made of a compound-interest table.

The second function, that of the checking account, is also taught by means of the bank forms. In considering the depositing of money, emphasis is laid upon the fact that the bank book is a receipt given to the depositor, and only the bank records the deposits therein. In withdrawing money the check is compared with a letter written to a friend who is taking care of some money for the writer of the letter; *e.g.*,

Cleveland, Ohio,
Sept. 4, 1920.

My dear John:

Upon his request, kindly pay to Arthur James five dollars (\$5.00) of the money you are keeping for me.

Yours truly,
(Signed) HARRY LONG.

Cleveland, Ohio,
The Detroit Av. Bank Sept. 4, 1920.
Pay to the order of Arthur James $\$5\frac{00}{100}$
Five and $\frac{00}{100}$ .... Dollars.
HARRY LONG.

The seriousness of an overdrawn account, as well as pride in being able to keep track of one's money, are incentives for accuracy in keeping the stubs. Situations are presented for practice which involve the making of several checks and the keeping of corresponding stubs.

The meaning of indorsement and the proper way to indorse a check are emphasized. The accommodation of being able to cash a check in a bank where one is favorably known, even if the check is drawing money from another bank, is brought to the attention of the pupils. The explanation of the adjustment of several banks' accounts through the clearing house is made by having the pupils act out the situations involved.

The class is divided into four or five bank groups, a name of a bank being assigned to each group. Each pupil makes out checks in favor of some one in the class, who represents a depositor in a different bank. When received, these checks must be properly indorsed and presented to the payee's bank

teller (a pupil in each bank group). When all have done this, it is supposed to be the end of the day. Then a representative from each bank goes to the clearing house (the back of the room) where the various banks settle with each other in the customary systematic way explained to them by the teacher. Returning to their bank groups, each representative has the checks which are now *vouchers*. These eventually reach the payer.

The third function, that of the loaning department, necessitates the study of a *promissory note*, which is compared with a letter conveying the needed information as was the check. The various kinds of securities: (1) personal, (2) real estate (mortgages), and (3) collateral are made known. What is meant by *foreclosing a mortgage* and what is meant by *equity* are explained. *Bank discount* of both noninterest-bearing and interest-bearing notes is studied in this connection, but the more customary *commercial draft* receives more attention.

The various ways of sending money out of town: stamps in a letter, for payment of very small amounts; postal money orders; bank checks and drafts; and telegraphing money are studied, along with their respective expense.

This is followed by the study of the different ways of sending things out of town: Parcel post, freight, express, trucks and aeroplanes, making, as accurately as possible, a comparison of the expenses connected with these methods.

Of course all of this work involves arithmetical computations, after the reasons for such are understood and toward the end of the course much time is spent in working problems which are in groups of five or six, having to do with closely interwoven business situations concerning one small group of people.

Care is taken to avoid such an arrangement of problems as would cause the whole set of examples to be wrong because of a mistake made in the first or even the second example. These examples afford splendid reviews and in such settings the ideas often become more clear to the pupils.

(The material up to this point may be easily completed in one semester. The following is a continuation to be studied the next semester.)

The next main topic is *taxes*, which is approached gradually through the following channels of thought:

(1) Personal expenses. How they are met.

(2) Family expenses. How they are met.

(3) Club expenses. How they are met.

(4) City expenses. How they are met.

The study of *personal budget* is made. Sensible percentages of the outlay of money are considered for the various apportionments. The advantages of allowances and the accurate keeping of personal accounts are emphasized.

Then the family expenses are considered. *Budgets for families* of various sizes with different incomes are studied, and in making budgets we have practical applications of per cents. Family accounts are kept, and in so far as is practicable the former topics are reviewed. Regular salaries versus incomes based upon commission are compared and ideas of thrift are presented.

The *club budget*, depending upon the type of club, affords an opportunity for constructive suggestions for worth-while clubs. The expenses, met by dues and assessments, bring out the simple phases of ratio and proportion. The *city budget* is always a revelation to the pupils. This study invariably awakens in their minds appreciation of the advantages of schools, libraries, paved streets, lighted streets, blessings which they had taken for granted before. They are then prepared for a fair way of paying for these advantages. The bulk of the money must come from the property owners. So they see a need for the valuation of the property, as ordinarily made by assessors. The money spent in paying for the expenses comes mainly from this assessed valuation. In figuring the part of the assessed valuation, which this tax represents, the *rate of taxation* is derived. Ratio and proportion are again made use of, in seeing what tax individual property owners should pay, as well as in seeing what part of the individual's tax goes to the county, state, city, school, etc., respectively. Other sources of income, such as *licenses, fines, etc.*, are touched upon.

The *county and state expenses*, met mainly by taxes and licenses, receive attention next. Meeting the *government's*

expenses completes the list in this topic. The stupendous expenses of the government are considered, the pupils bringing to class as much of such information as possible. The main source of the government's income, which is the *custom duties*, is then studied from the viewpoint of their being a protection to our industries. The reason for a *free list* is shown, and the *ad valorem* and *specific duties*, along with *invoices* of imported goods, present good material for arithmetical work. The simplest phases of the *income tax* are also studied in this connection.

The chain of topics is unbroken when the subject of *insurance* is introduced, as it is another activity concerning a group of many people.

Property insurance is studied with the following ideas in mind:

(1) In insuring property, protection is being paid for.

(2) The greater the protection required, the higher the rate should be.

(3) What increases the need for protection? Consider the location of the property involved, its condition, etc.

(4) What decreases the need for protection? The pupils bring to class as many *insurance policies* as possible. These are carefully examined, and the terms used in connection with insurance are introduced and learned by being used. The pupils are expected to gather information concerning rates of insurance on stores of various kinds, dwellings, moving picture places, etc. These rates are discussed and used in figuring premiums to be paid for places insured for various sums.

Insurance against loss by fire, burglary, tornadoes, etc., are thus studied before life insurance. *Life insurance* is considered by means of comparing a person's life with his property. The risks differ in what way? Why is a physical examination necessary? Why should the rate depend upon one's state of health? What is a life worth? A man may be insured for any amount for which he can pay. Why?

The four principal kinds of life insurance: (1) Ordinary, (2) Limited, (3) Endowment and (4) Term, are discussed.

Using real policies, problems based upon the face, rate and premium are made up.

The next series of lessons which completes this outline, while taking up new ideas, review most thoroughly every one of the preceding topics.

The class is led to follow the gradual growth of a business from its start, when it is owned by an individual, until it develops into an incorporated stock company. The pupils choose the type of business they wish to consider from a list of several selected by the teacher. The teacher is guided in making her selections by the idea of making use of as many of the previously studied business activities as possible. The sale of imported goods, for instance, needs the understanding of: profit and loss, commercial discount, commission, banking, sending money, sending goods by freight or express, custom duties and insurance.

As an illustration let us follow a class which has chosen the business of selling toys.

One man (let us call him Mr. A. J. Reynolds) owns the business. The pupils are expected to do research work, finding out reasonable rents for stores, and other overhead expenses, as well as the approximate cost of toys. The necessary capital is then decided upon.

Mr. Reynolds finds it necessary to increase his capital, so he borrows a few hundred dollars from the bank, offering security therefor.

In the progress of the development of the business, problems embodying the following ideas are worked: wholesale price invoices, custom duties, expense of sending money, expense of transportation, gross gain, net gain, rate income from the investment.

The business develops nicely. Special sales attract customers and commercial discount again appears naturally. Salesmen are engaged on the commission basis and their incomes are figured.

More capital means more business, so a *partnership* is formed. This makes it necessary to know how to divide profit or loss equitably. Splendid material for the study of ratio and proportion appears in this connection.

Realizing the advantage of a larger capital they decide to form a *stock company*. By careful guiding and helpful sug-

gestions the following terms are understood and *then* the names are given: stock, shares, becoming incorporated, preferred stock, common stock, dividends. Reasons for wanting to buy stock from a stockholder come up, as well as reasons for a stockholder's desire to sell his stock. Then reasons for increased or decreasd values of stock are realized. Again use is made of the newspaper. This time the stock quotations are studied. The meanings of par value, selling stock above or below par, rate of income from an investment are brought out. A stock broker's manual, aiding in deciphering the various stock abbreviations and giving information concerning the companies, is used along with the newspaper.

The study of these reports in newspapers of several successive days shows how the values fluctuate, and lessons concerning the uncertainty of stock values are pointed out. The explanation of "bulls," "bears" and "the shorn lambs" make interesting, but not very attractive, the investment of one's earning's in stock, especially when it is not listed.

Our company wants to raise money to pay a debt or to buy a store or for some other legitimate reason. Hence it borrows money, giving its promissory note to the loaner. This note is a *bond*. Taking care of a large loan by issuing several *bonds* is explained. The difference between a *registered bond* and a *coupon bond* is taught. The manner of receiving the interest is shown in each case. *Municipal bonds* and *government bonds* are studied also. The newspapers show that bonds also are bought and sold at fluctuating prices.

The necessity for knowing an investor's *rate of income* is brought out and other investments are used as a source of study.

The series of lessons following "insurance" is of a character such that problems of a cumulative sort are used almost exclusively, causing independent thinking and introducing the pupil to natural business situations.

All of this material may be used to advantage in carrying out the fundamental aims of mathematical teaching as announced by the National Committee on Mathematical Requirements. In its preliminary report of our Junior High School Mathematics, at the bottom of page 4, we read, "The primary

purpose of the teaching of mathematics should be to develop those powers of understanding and analyzing the interdependence of quantities . . . which are necessary to a better understanding of life and of the universe about us, and to develop those habits of thinking which will make those powers effective in the life of the individual."

In planning this outline, the desire has been to eliminate as much of the artificial as possible, and to develop the understanding of life situations in the most natural way.

It is realized that this outline is devoted exclusively to the *one* phase of socialized arithmetic which arises in business situations, while there are other lines of work equally rich in useful arithmetical material. The arithmetic of science, both theoretical and applied, as well as the computations necessary in technology and industrial life, are sadly neglected in our schools. It is hoped that these other branches may, with the expansion of the curriculum, receive adequate attention, not, however, at the expense of the business arithmetic with which everyone should be very familiar.

The sole aim of this paper is to call attention to an *organic* presentation of worth-while business problems which has been thoroughly tested in a large number of classes.

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MATHEMATICS IN THE HORACE MANN SCHOOL FOR BOYS.

A teacher of English who is notoriously deficient in mathematical interest stopped me in the corridor one day and said, "What in time is X doing in his classes? Every time I pass his door those boys arouse my curiosity. For they are doing things that I never had when I went to school; things that look interesting!"

Well, what were those boys doing? Perhaps he caught a glimpse of some youthful "seventh graders" trailing a tape measure around the room, or squinting along a rod out of the window, or learning to use a pantograph. Or maybe he saw a class of ninth-grade boys being initiated into the mysteries of their fascinating new slide rules, anxiously comparing the setting of their rules with the setting on the huge slide rule hanging on the wall in front of them. Or perhaps he peeked in at times, only to find the rooom deserted. And then he remembered having seen the youngsters exuberantly leaving the building armed with stakes, tapes, and some funny-looking homemade instruments which I had to tell him we playfully called altazimuths to give them some standing in the community; or he had just passed the seniors on their way out to give the transit and level rod an airing.

What is it all about? Has X caught the project fever? Far from it. Ordinarily you will find him busy teaching the good old-fashioned mathematics, with the boys doing their full share of thinking, and of hard work without frills. But at the right time he can inject new interest into the work.

On what is this new interest built? Three fundamental aims: (1) To develop the curiosity of the boys who come to us; (2) to give them that knowledge and that appreciation of their surroundings which may be expected of cultured persons; and (3) to increase their ability to serve society. And that they may serve society more efficiently, we of the mathematics department hope that they will acquire not only the content of

certain particular subjects and the special technique pertaining thereto, but that they will also learn to think accurately, to work thoroughly, and to attack intelligently and successfully the problems which confront them.

Before proceeding to outline the course of study in mathematics which we use in attaining our aims, a word or two as to the organization of the school and of the department will not be amiss. The school is a six-year high school, in which in fact, though not in name, the first three years form a junior high school and the last three form a senior high school. The motive underlying this actual division of the school is our desire to differentiate the younger boys from the older boys. The junior school periods of 55 minutes, the consequent opportunity to train the boys in Grades VII., VIII., and IX. in proper habits of study, and the necessity of giving them shorter home-work assignments, all call for a treatment essentially different from that accorded the older boys, who meet in class for only 40 minutes and are expected to cover much longer assignments working by themselves.

Each teacher in the department of mathematics has classes in both the junior and senior high schools. He is fully acquainted with the aims and policy of the department and knows intimately what we collectively are trying to do in the upper and lower schools. The deeper understanding and broader vision which he is thus enabled to bring to each of his classes results in a coordination of all the different courses into one continuous whole.

The course of study in mathematics is as follows:

Junior High School (Required).

Grade VII. Arithmetic and Intuitional Geometry.

1. Twelve weeks: Systematic treatment of fractions and decimals; percentage. Practical measurements; drawing to scale; the decimally divided foot.
2. Seventeen weeks: Intuitional geometry and mechanical drawing, with special attention to practical problems.
3. Five weeks: Introduction to ratio and proportion and to square root in connection with elementary surveying work.

Grade VIII. Algebra and Arithmetic.

1. First term: Solving problems involving the formula, the equation, simple algebraic operations, negative numbers, and graphs.

2. Second term: Applications of algebraic methods to the solution of problems in arithmetic, with special emphasis upon those problems in percentage, interest, and mensuration which arise in the home, the office, and the school.

Grade IX. Algebra and Computation.

1. Special products and factors. Fractions and fractional equations. Simultaneous equations. Introduction to the laws of exponents, to radicals, and to numerical quadratic equations. Much emphasis laid on the solution of graded problems.
2. Extended course in computation by means of logarithms, slide rule, tables, graphs. Application to advanced arithmetic, to the mensuration of plane and solid figures, and to numerical trigonometry. Approximations and checks.

SENIOR HIGH SCHOOL (ELECTIVE).

Grade X. Algebra and Geometry.

1. The laws of exponents; radicals. Quadratic equations. Simultaneous equations. Progressions. Binominal theorem. Proportion, variation; literal equations; equations of any degree. Review of algebra, including abstract problems.
2. Plane geometry. *Book I.* and review; original exercises. Part of *Book II.*

Grade XI. Geometry.

Plane geometry, *Books II.-V.* Intuitional solid geometry Review of plane geometry. Many original exercises throughout the course.

Grade XII. Geometry; Trigonometry; College Algebra.

1. First term. Solid geometry.
2. Second term. Plane trigonometry. Use of the transit.
3. Second term. College algebra. Elementary notions of the calculus. Slope of a curve; rate of change of a variable; the derivative. Differentiation of the simplest algebraic functions. Applications to geometry and physics.

Each course meets five times a week.

Paralleling the last four years of this course of study is a second course of study in which algebra and then plane geometry are completed in turn, a year and a half being devoted to each subject. Following these comes the work of the senior year in solid geometry, trigonometry, and college algebra as outlined above. This second course of study serves two purposes: it accommodates in one set of sections those boys who have completed elsewhere the eight grades of the elementary school and wish to enter our ninth grade; and it makes possible the formation of other special sections for groups of boys

whose rate of progress should differ from that of the majority.

The following are some of the considerations which led us to adopt the course of study as given above for the six-year high school. Since abstract reasoning comes with maturity, such subjects as literal equations, the theory of exponents, and most of the abstract problems of algebra have been placed as late in the course of study as practicable. For a similar reason, demonstrative geometry, treated as a course in formal logic and in precision of statement, is begun in the middle of Grade X. Moreover, we devote a year and a half to plane geometry. As a result of this only a very few boys fail to pass in it: and, in addition, since there is ample time to do many original propositions, properly graded in difficulty, most of the students acquire a habit of straight thinking and of clear and accurate statement which will stand them in good stead in other fields and which only a few would be able to acquire if the subject were completed in a year.

As outlined, perhaps this course of study seems as dull as any other, but—

In Grade VII. we are confronted each year in September with a group of new boys from different schools. A barren review of the fundamental operations, in which they are not uniformly skilled, would be uninspiring to teacher and pupil. To obviate this we start with mensurational work, indoors and outdoors; and then through problems which arise from this work we effect a review of the fundamental operations. From the very beginning the pupils are introduced to the general notion of significant figures. Following this comes percentage, which with its numerous applications occupies the boys' attention until Christmas.

The first four months of the new year are devoted to intuitional geometry, with elementary mechanical drawing. After which the remaining months in the spring are given over to elementary surveying work done outdoors in connection with the subjects of proportion and square root. In this course the boys learn how to measure lengths and angles, directly and indirectly, and how to construct and use a simple altazimuth.

Again, the second term of Grade IX. has seemed to us a fitting place to give a course in the use of logarithm tables, the

slide rule, other devices for rapid computation such as adjacent scales and alignment charts, with many applications to advanced arithmetic. Compound interest and present value, statistics, variation and proportion, and mensuration offer a wide field in which to exercise these new aids to calculation. The question of significant figures colors the work throughout. Still further practice is derived from numerous problems in numerical trigonometry, including the logarithmic solution of oblique triangles.

The curiosity which caused the English teacher to pause outside the door is of course much stronger with the boys in class, who are led over the necessary drudgery inherent in the best mathematics teaching by the impetus which these high points of practical interest impart.

Ralph Beatley.

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VALUES IN HIGH SCHOOL MATHEMATICS.[1]

There is no need of telling those here present that mathematics is in a rather precarious state at the present time in the field of education; the educational iconoclast has declared that we shall teach what the child likes and not what the child needs: and as a result we have a declaration of purpose to practically eliminate mathematics from the subjects of study in our schools. True, a pupil may elect mathematics if he or she wish, but I believe that when mathematics ceases to be obligatory in the high school, and a diploma may be had with no knowledge of mathematics, we may look to a very near future when comparatively few will be studying mathematics.

Since the sciences, physics and chemistry, rely so much upon at least a fundamental knowledge of algebra, geometry and even trigonometry, they too will suffer from the passing of mathematics.

It seemed to me a very apt time to take up this question before this convention and endeavor to show the value of mathematics in a high-school course, so that we may go forth determined to fight for the retention or, mayhap, re-introduction of mathematics as a required subject in every high-school curriculum. The National Committee on Mathematics Requirements would seem to desire that two years of mathematics be required for a high-school diploma, and we may align ourselves with them and see what would be the value of these two years of required mathematics.

I think we are pretty much of one mind in regard to the value of mathematics in education. Who can judge more surely of the educational value of a subject than those who have spent years in studying and teaching that branch? The value of mathematics in a high-school course and the making it obligatory for all pupils would seem to me to hinge upon the more fundamental point as to what is the chief or primary purpose of a high-school course. If it be to give "culture,"

[1] Paper read before Central Association of Science and Mathematics Teachers, 1920.

the cultural subjects should and must predominate, and only so much of "informational" studies will be admitted as is absolutely necessary. Moreover, even these must be so taught as to bring their own proper "cultural effects" to a maximum. Here I might pause to make clear my terms and avoid possible misunderstanding. By the "cultural" value of a subject, I mean the development of broad views and power of generalizing; arousing of mental alertness and eagerness; development of initiative and self-reliance in forming judgments and meeting difficulties. The fine arts in particular develop correct taste and appreciation of beauty. "Cultural" effects thus include all that tends to modify a student's point of view and habitual way of thinking.

The "disciplinary" value of a subject consists in the training of memory, drill, development of habits of accuracy, neatness, persistence, application, and logical (deductive) thinking.

Thirdly, the "informational" value is the acquirement of knowledge of useful facts, laws, events, theories; as in geography and physics or history taught by recitation from the text-book.

These values often overlap. But modern pedagogy has outlined these values and we take them as put. I need hardly state that the universal belief and conviction of educators is that the prime purpose of a high school is to give culture, not information. Discipline and information hold or should hold secondary places in the aim of a high-school course. True, there are institutions of high-school rank in which information and not culture is paramount; and even here the ideal striven after is to vitalize utilitarian work in every way and make it, so far as may be, cultural and power-giving. An immediate result of this view of the high school is that it becomes necessary to define the precise cultural effect inherent in each branch of the curriculum, and to lay down methods of teaching, which shall make it contribute this effect in the maximum degree, without losing aught of its informational and disciplinary value. And if mathematics be made an obligatory study, at least for one year, in every course in the high school, then we must show that algebra and geometry contribute to the primary end of a high school in such wise

that, to dispense with them in favor of any other branch, would be to defeat the very purpose of high-school education. It is for us to show that algebra and geometry do contribute chiefly to culture as high-school subjects.

Not long ago, there were those who thought and maintained that the "disciplinary" effect was chiefly, if not the only thing worth while in mathematics; emphasis was laid on the drill, the training in deduction and the habit of rigid logic. Undoubtedly, this is one of the important values of mathematics as a high-school study, but much more can be done with it than that. Under one teacher an algebra or geometry class is interested and enthusiastic, and grows in ability and self-reliance each day; under another teacher, the algebra hour is dull and wearisome, the pupils' minds are deadened by drudgery in an alphabetical treadmill, and they learn chiefly this: to abhor for the rest of their lives the very name of mathematics. In such a class the "disciplinary" value of algebra and geometry is being emphasized to the neglect of the "cultural." I feel justified in stating that very few boys or girls, who enter high school, have a natural want of aptitude for mathematics; they do not naturally hate it; but if it be made a bore to them, they will grow to abhor it. Would it not seem that the method or mistaken viewpoint taken in teaching high-school mathematics, is to be rejected and criticized as of no value, rather than the subject itself? The best food in the world can be spoiled by a poor cook.

Mathematics can be so taught as to arouse interest and leave pleasant memories of the time spent in its study. Mr. C. Godfrey wrote, some years ago, in answer to one who styled mathematics a blighting shadow of youthful days: "High manipulative dexterity is not the purpose sought; this belongs to a specialist." Just so, a pianist must spend an astonishing number of hours at mere mechanical technique. But the average man finds no occasion for algebraic calculation, and if he does, the opening is declined in spite of the heavy work that darkened his school days—perhaps because of it. But we are profoundly convinced that the general mathematical ideas and modes of thought wrought into the mind by a suitable course of instruction are of permanent cultural value; these are a

necessary element of a liberal education; whereas dexterity in manipulation is a specialized, technical accomplishment, like glass-blowing, or Latin verses, or playing piano, or shorthand —all very excellent but not necessary to liberal education.

Objection is made that we sacrifice mental and moral discipline by removing merely mechanical drudgery, and giving the pupil *new ideas* instead; but experience refutes this. We still have drudgery but it is vivified by a purpose and a sense of progress. Just as a man walking in a treadmill gets mechanical exercise, so a man walking in the park gets the same, —but, one is, moreover, refreshed and exhilarated mentally, while the other is only wearied and dulled after his purposeless toil.

Let us now turn to the specific subjects our report deals with for the first years of high school,—algebra and geometry, and see what is their cultural effect, their importance.

First, algebra, if exacted strictly—no matter how taught—will produce a "disciplinary" effect on the character, in so far as the pupil has to drive himself to do what he does not like; he learns self-control and the faculties are drilled. These effects, being common to *all* studies whatever, and even to mere physical labor, are not of present interest. If algebra did only this for a pupil, I see no reason why it should occupy a place in the high-school curriculum above any other branch.

Algebra always gives, in the second place, a certain amount of information to a pupil, the utilitarian value of which, in his future life, is negligible. Some few pupils will adopt a career in which algebra is turned to use, but these are hopelessly outnumbered. We may safely put down zero as the informational value of algebra to the average pupil. Here again no reason for making all take algebra in their high-school course.

Algebra, therefore, if it is to give anything of lasting value to a pupil, must give it in virtue of the "manner in which it is taught"; the pupil must be "changed" in some way by this study and get from it a certain power, inquisitiveness, self-reliance, directness of thought, which he had not before and which he could not readily get otherwise. In other words, algebra contributes to culture, to the primary end of a high-school course and directly, for:

1. A pupil in solving exercises or problems learns to muster his resources and use them independently; he acquires self-confidence in attacking or circumventing difficulties, learns to think correctly, learns how to test and check his conclusions when in doubt; he gains faith in his own powers and his originality develops.

2. The "symbols" form a new language very condensed, and idiomatic; the pupil learns to translate to and from these and again acquires self-confidence and power.

3. The habit of "generalizing" gradually grows upon him and gives him broader views and the power of grasping and subordinating details.

4. Incidentally the pupil is acquiring habits of mental accuracy and of concentration of mind upon one point.

5. The "sense of satisfaction" in discovering new things or solving hard problems makes him eager to try his powers on other difficulties independently; it becomes a game in which he is intensely interested and his self-reliance continually grows day by day.

This widening of the student's grasp of universal ideas, of learning to include many differing cases under one heading, namely, the habit of generalizing, is one of those habits which is *transferable* to other lines of thought. We all know what a debate has been raging for years in pedagogical text-books on this point of "transferability of habits." Algebra, if taught inductively, not didactically, produces this effect in the pupil.

The "informational" value of geometry is, in most cases, nil. The "cultural" effect is twofold in its origin; we have:

1. "Problem work, original exercises," the result of which is to cultivate the student's self-confidence, independence, skill and resourcefulness. He learns to separate the essential from what is irrelevant; he learns to construct continuous chains of reasoning; and, finally, to be precise and accurate in thought and speech.

2. He comes in contact continually with explicit syllogistic reasoning, the beneficent effect of which is far reaching. Above all, geometry arouses and brings into play the "detective-instinct." A few clues are afforded in the statement: the pupil must now be a sort of Sherlock Holmes and figure out

the whereabouts of the unknown answer, and rig up an air-tight proof for it. This faculty is again, transferable; so that a year of detective work in geometry will make a pupil more expert in using detective-methods in law or medicine or any other line of work, even commercial.

The importance, then, of algebra and geometry in a high-school course would seem to be paramount. They tend to promote directly and effectively the primary end of the high school—give culture to the pupil, and they would, therefore, appear indispensable to the training of every pupil.

Should we not, then, exert ourselves to bring this importance of mathematics in the education of the young before the great public; to enlighten parents who may fail to realize the value of mathematics and whose duty it is to see that their education is properly given? Should the election of mathematics be left to the young pupil, how many thousands the country over will be governed by ease of acquirement, and choose some branch which makes for little compared to the development mathematics surely gives?

In St. Louis, mathematics is an elective in five out of the seven courses open to high-school pupils. We are banded together in the Mathematics Club of High School Teachers of Mathematics and are waging a determined battle to bring mathematics back into the course as a required subject for at least two years. Business men have already written letters to the newspapers urging the need of mathematics for every high-school student, and the light of publicity is being focused upon the loose arrangement whereby the inexperienced pupil is allowed to choose what often taxes the judgment of a mature mind—the election of subjects which best fit him for future life.

It is our hope that the members here gathered from many schools and cities will see fit to approve our determination and spread to many cities what we wish to bring about in St. Louis,—that mathematics regain its place in a high-school course, as a required subject, because it is a most important element in the proper carrying out of the end and purpose of high-school training.

REV. W. J. RYAN.

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ST. LOUIS, MO.

THE TEACHING OF LOCUS PROBLEMS IN ELEMENTARY GEOMETRY.*

Locus problems constitute a part of geometry usually dreaded alike by pupil and teacher. The pupil may be confident of his ability to demonstrate all the propositions and may be master of the facts and processes that are the basis of numerical work, yet the area tenanted by locus problems will appear to him a trackless and endless morass in which are lurking perils innumerable. The teacher is in doubt as to which of the unpromising paths to choose, and, when once plunged into the maze, where he may safely halt and reconnoiter. The travail of such repeated experiences has brought forth the outline below.

Before turning to the outline in detail, a warning should be uttered against taking up locus problems too early. They are not adapted to the beginner. There is a general opinion today that demonstrative geometry may well be preceded by considerable work in constructive or intuitional geometry, and this because rigorous reasoning is not a function of the immature mind. Similarly the idea of the locus is a still more highly generalized conception, and should bide its time. If a course in Plane Geometry is to consist of a beginning year followed by a review course, little if any locus work should be done the first year. If all is to be crowded into one year, the locus element should be held in abeyance until the latter half of the course. Much time and vexation is saved by such delay.

Moreover, a scheme that delays the introduction of the locus idea until the class is approaching the end of its course in geometry affords an opportunity to use it for review, an incidental use to which it is particularly well adapted. The ordinary locus problem involves no new principles or facts. It merely re-groups old facts, and faces them from a new point of view. From a pedagogical standpoint, it occupies about

* Delivered before the Association of Teachers of Mathematics in New England, December 4, 1920.

the same place that the topic of inverse functions does in trigonometry. Both consist of old facts under new statements, and both serve alike for effective review. If these topics find a place in a course the moment the material for them is at hand, their function as reviews and summaries is largely lost.

The outline below, the outgrowth, as has been said, of actual class-room work, is constructed from the practical rather than from the theoretical view. It seeks to lead the pupil in two respect. First, it shows him how to formulate the knowledge he has on a stated problem in the form and language that the locus conception has by convention assumed; that is, to tell him what he ought to say, in what form he should arrange what he says, and when he has said enough. It aims to give him confidence when he finishes a discussion that he has covered the ground sufficiently, and also to develop in him a check against "running to useless words." Second, the outline will aid the pupil to discover systematically such facts as he may need for the basis of his conclusions. In short, it supplies a definite working rule, a common ground between teacher and pupil.

It may be added, moreover, that the method of treatment of the outline is determined by the limitations of elementary mathematics rather than by any ulterior aim. Courses in college may often, even in the freshman year, find it advisable to shift the emphasis of the point of view. There is no real difficulty in this if the student brings to his new task a clear and fundamental idea of the nature of a locus. The best preparation for advanced loci is a mastery of elementary loci.

The material in larger type is intended to be placed before the pupil. He should become familiar with the heads and subheads. The supplementary explanatory matter in the small type is primarily for the teacher.

Outline for Handling Locus Problems.

A. *Definition.*

A locus is the goemetrical figure occupied by a series of points (or lines) that fulfill a given condition.

The pupil's attention may well be directed to the fact that the word locus means in geometry just what it has meant in Latin: namely, the *place* where

an object is. The words "geometrical figure" cover all the geometrical elements, line, point, surface, solid. The phrase, "series of points" suggests correlation, and in a way, continuity. The words in parenthesis "or lines" are not likely to be needed at first, not until solid geometry is reached. It seems that the conception of a locus or a figure is better adapted to beginners than that of the path of a moving point.

Many teachers will at once miss in the definition a final qualifying phrase, "and no other points." This is purposely omitted. There is, of course, no thought of suggesting that the locus should not be restricted to its minimum limits. It is believed, however, that the strict meaning of the word "occupied" itself is sufficiently restrictive. Such parts of an assumed locus that are not definitely *occupied* by any of the points in question will be ruled out of the application by a strict interpretation of this verb. There is a marked advantage in having any definition concise, and also in avoiding, as far as may be, the distractions of qualifying phrases. If it be feared that the pupil will not sufficiently assimilate the restrictive element from the wording of the definition, he will certainly receive it from Part III. in the following discussion of loci.

B. Discussion of Simple Loci.

1. The discussion of a locus problem consists of *four stages*. *Discover* and *state* accurately what the locus called for is. This step is unnecessary if the problem as given states itself the locus in question.

(*a*) To *discover* a locus, plot a few points, wisely scattered, that satisfy the given condition, and from these forecast the complete locus.

This process of systematically proceeding to discover a locus is often wholly or largely unnecessary, inasmuch as in many cases it is evident at once or almost at once what the locus in general terms is. This process of discovery has much about it akin to developing the negative of an unknown photograph, in which process the strong features come out gradually until soon the subject of the plate is evident. That which is meant by the phrase "wisely scattered" hardly needs explanation. The pupil should be taught to place some of the points he is using toward the extreme positions, and especially in peculiar ones. The word "forecast" suggests the proper attitude of the pupil's mind during this process.

(*b*) To *state* what a locus is, give attention to (1) what it is, (2) where it is, (3) how big it is. As to (1), it may be a point, a line, a surface, or even a solid, but in plane geometry is usually a line, straight or curved.

This power of statement is really the heart of any locus problem, and

cannot be too much emphasized. If the pupil considers in turn, in forming his statement, the three points just mentioned, he is not likely to omit any essential part of the statement. It is worded here in homely terms in order the more to impress the pupil. The first point names the locus, the second, locates it, the third, measures it. In loci of indefinite extent, such as the perpendicular bisector of a line, there is of course no need of the third point.

The pupil in plane geometry, in its present limits, should be reminded that if it is apparent that his locus is a line that is not a straight line, it must be the arc of a circle, inasmuch as no other curve is treated in plane geometry. This does not forbid the construction by the pupil from the proper data a parabola, simply as an illustration in discovering a locus which he is pretty sure not to know. He would, however, be expected to lack the ability to name or describe this.

2. Prove that any point in the assumed locus satisfies the given condition.

For this purpose, it is usually well not to employ the tentative figure that has been used in discovering what the locus is, but to draw a new figure containing the supposed locus. Any point in this, definitely selected as a sample of them all, is then to be proved to fulfill the condition. Sometimes a single preceding proposition will cover this; at other times more or less original proof is required.

3. Prove that any point that is not in the assumed locus does not satisfy the given condition.

Sometimes it is simpler to show instead that any point that satisfies the condition lies in the assumed locus. This step, (3), is sometimes practically self-evident from step (2), and then scarcely requires proof.

The pupil should be required at first, however, to prove this step rigorously, even in the simpler cases. For example, if the question is that of the locus of a point 2 inches from a given point, he should take a point outside the circle and show that its distance from the point is greater than 2 inches, because the whole is greater than any of its parts; and similarly, with a point inside the circle.

4. Summary, showing how the locus has been proved.

This step is never absolutely necessary, but it has to the previous discussion what the final summary of a debate has to the argument, what the traditional Q. E. D. has to a demonstration, and perhaps what the Amen has to a song or prayer. It leaves us with the thought that the job is done.

C. Intersecting (or Compound) Loci.

1. The locus of points that satisfy two or more given conditions is found by constructing the locus for each and noting the intersections.

The pupil should be led to construct each of his loci quite independently of the other, merely superposing them on each other.

2. Special cases that modify part of the intersections must be particularly considered. The number of points of intersection may be zero, one, two, three, four, . . . or infinite. These special cases depend on (*a*) position, (*b*) dimension.

Too much effort can hardly be given to develop the attention and ingenuity of the pupil in detecting the peculiar situations that modify the intersections. In many cases it is adequate to let these be shown by the pupil by a drawing accompanied by a short, descriptive phrase, rather than by an elaborate statement.

3. Among the simple loci commonly met in these intersections, and with which as fundamentals the pupil should be familiar, are:

(*a*) All points at a given distance from a given point; from a given line; from a given circle.

(*b*) All points equidistant from 2 points; from 2 intersecting lines; from 2 parallel lines.

(*c*) All centers of circles of a given radius through a given point; passing through 2 given points.

(*d*) All centers of circles tangent to a given line at a given point; to a given circle at a given point; to 2 parallel lines; to 2 intersecting lines.

It is not the purpose here to give any list of locus problems. It may be suggested, however, that while the number is somewhat limited, and while firstclass locus problems are always at a premium, yet there exists a considerable quantity of ore to be worked over. The mines that exist under the topics of measurement, of proportional lines, and of areas should not be neglected. It is particularly unfortunate to attach the opening work on loci so much to points that are equidistant from certain things as to forever connect in the mind of the pupil a locus with the thought of equal distance. This is very often done through the shaping influence of the thought of the two locus problems that the pupil first meets in the traditional order, the locus of points equidistant from the ends of a line, and the locus of points equidistant from the sides of an angle.

As material for the first work on loci in plane geometry, the following propositions are especially useful:

(*a*) The locus of points equidistant from two given points.

(*b*) The locus of points at a given distance from a given point.

(*c*) The locus of points at a given distance from a given line, (1) of indefinite length, (2) of definite length.

(*d*) The locus of the mid-points of lines drawn from a point to a given line.

(*e*) The locus of the vertices of the right angles of right triangles drawn on a given hypotenuse.

(*f*) The locus of the mid-point of chords drawn in a circle from a given point on the circumference.

(*g*) The locus of points equidistant from a given point and a given line. (Not to be stated.)

More harm than good is usually done by the setting of locus problems that are nothing but puzzles, and that fail to illustrate important geometrical principles. They may serve to interest the pupil of unusual ability, but to the ordinary student they impart distaste of the whole subject.

A somewhat extraneous thought connected with the matter of loci is that the statement of a locus should seldom be used as a reason in demonstrative geometry. The reason on which a statement depends is more effective in the form of a proposition than when it is cast in the form of a locus. In certain theorems, failure to observe this suggestion leads to absolute worthlessness of proof. For example, if in the proposition that the bisectors of the angles of a triangle are concurrent, the pupil gives as a reason "the locus of points equidistant from the sides of an angle is the bisector of the angle," the application of this is exactly opposite in the later step of the proposition from what it is in the former steps, and consequently the statement can lead to nothing but slovenliness and inaccuracy on the part of the pupil.

The use of the methods of attack and procedure developed in the preceding paragraphs will aid, it is hoped, in bringing from the limbo of the mysterious a branch of geometrical work that is alike illuminating as a new outlook on its own subject matter, and valuable as an opening into the great mathematical avenues beyond into which some of our secondary school pupils are to enter.

FRED D. ALDRICH.

WORCESTER ACADEMY,
WORCESTER, MASS.

COMMUNICATIONS.

I.

To the Members of the National Council of Mathematics Teachers:

I deeply appreciate the honor which you bestowed upon me when you elected me president of your organization, and I am also mindful of the responsibility which this office carries. It is my hope that we may further the organization and lay plans for the improvement of the teaching of mathematics as a means of training for citizenship. In this I earnestly solicit the cooperation of mathematics teachers throughout the country.

It is too early to state definitely the plans for the work of the years, but they will doubtless include the following:

1. The careful consideration of suggestions made by any who are interested in the teaching of mathematics.

2. Furthering the work of the National Committee under the leadership of Dr. J. W. Young by trying to have its findings used as the basis for the courses of study in our Junior and Senior High Schools.

3. Placing in the hands of teachers the detailed material without which the work of the National Committee cannot succeed to the fullest.

4. An effort to determine the best method to present the work outlined by the National Committee.

5. An effort to continually improve the Mathematics Teacher and to extend the services rendered by it.

6. The extension of the membership of the National Council to include, as nearly as possible, all teachers of mathematics throughout the country.

7. The appointment of a representative in each state who shall be responsible for the work of the Council in his state.

J. H. Minnick.

University of Pennsylvania.

To the Editor of the Mathematics Teacher:

In the twentieth annual report (1920) of the Secretary of the College Entrance Examination Board appears the following chart:

This is a highly significant chart for teachers of algebra in the Eastern States. A standard which exhibits a fluctuation in the number of successful candidates of from less than 40 per cent. to more than 70 per cent. in a period of three years is obviously no standard at all. And yet it is the standard. Are these fluctuations with their untoward effects upon the schools necessary?

Chart VII. Elementary Algebra. The solid line shows the percentage of candidates whose books were rated 60–100. Dotted line shows for all books written at all ordinary examinations the percentage rated 60–100.

In answer to this question we can maintain with reason that the fluctuations are due both to the examinations and to the method of rating. At present the group of successful candidates in any year is composed of those who can answer correctly 60 per cent. of a paper subject to the definition of the value of parts of the paper as determined by the readers for that year. To maintain a standard under this system would require that the papers be of a standard degree of difficulty

and that the definition of relative values of parts of the papers bear a reasonable relation to their actual difficulty. These statements are of course predicated upon the assumption that the preparation of the candidates is relatively uniform from year to year. The dotted line in the chart shows this to be a reasonable assumption for it is unlikely that the preparation in mathematics would fluctuate violently while that in the other subjects did not. If it be contended that the dotted line shows nothing but the smoothing out process of averaging we can still maintain with reason that it is unlikely in the nature of things that there should be such *extreme* variations in view of the inertia of large numbers and the country-wide nature of the representation.

In view of the efforts made by the Board to remedy the situation by tinkering with the examinations the chart is abundant evidence that there is nothing much to be hoped for in that direction. The machinery of the Board offers no possibility of standardization. If there is nothing to be hoped for from the examinations is there anything which can be done to relieve the situation?

The writer is of the opinion that the situation could be alleviated by reporting the results of the examinations by groups under an annual definition of the groups which would insure reasonable uniformity.

"To pass" is a definite act. "A passing percentage" is a snare and a delusion, for experience shows that it is not a constant but a variable with a wide range. When seventy-five per cent. are successful the colleges complain of poor material. When less than forty per cent. are successful the schools know that adequately prepared candidates are eliminated.

Howard F. Hart.

Montclair, N. J.

DISCUSSION.

The Editor is not satisfied with the comparatively small number who are taking part in these discussions. Any good teacher must have valuable ideas about subject matter, methods, and other mooted points. It is largely by the interchange of such ideas that we grow and improve our work. Send them to the Discussion Editor and help to make this department really helpful.

In particular, send in suggestions on the following question: "What are the few really important problems that teachers of mathematics are meeting?"

If we can define those problems, bring them into the light of day, and then concentrate on them, we shall probably make them shrink very decidedly. And we can keep on discussing our minor questions at the same time.

Suggested Topics for Discussion.

Class Tests. How often shall they be given? How marked? Should there be tests where absolute perfection is required for credit on a question? How can the questions be chosen so as to give most value to the test? Should the test be too long for the average child to finish?

Text Books. How can one best examine a test book to find out whether it is suitable for use in his classes? What use can be made of texts as supplementary material.

Answer to Q. 6—Is it advisable to have children check the solution of an equation when the check is far more difficult than the solution?

While it is of doubtful advantage to require pupils to do any great amount of checking when the practice involved is not worth the time used, they should occasionally check such a solution even though the check is more difficult than the solution. The realization that even such a complicated operation can be carried through to a successful conclusion and will prove the work correct is worth the time spent.

But should there be many questions where the check is so difficult?

Eugene R. Smith,
Editor.

NEWS AND NOTES.

Dr. John H. Minnick was elected President of the National Council of Teachers of Mathematics at the annual meeting at Atlantic City. Dr. Minnick has been unusually active in secondary school mathematics. He is now engaged in training high school teachers in mathematics in the University of Pennsylvania. Many readers will recall Dr. Minnick's Tests of Abilities in Geometry, a scientific monograph on the nature of the abilities which are involved in proving a proposition in geometry.

Announcement has come to the Mathematics Teacher that the Harvard School of Education and Teachers College will offer summer courses in the teaching of mathematics which will consist both of theoretical discussions and of class room observation or demonstration. Daily observation in these demonstration classes proves most stimulating to teachers who are enrolled in professional courses. Mr. Ralph Beatley, head of mathematics in the Horace Mann School for Boys, will conduct the courses in Harvard. Mr. Raleigh Schorling, principal of the high school grades of The Lincoln School, and Mr. C. B. Walsh, principal of the Friends' Central School of Philadelphia, will conduct the courses in Teachers College.

Dr. David Eugene Smith will publish in early issues of the Mathematics Teacher a series of historical articles on problems which bear directly on high-school mathematics. He is now publishing a series of such articles in the *Bulletin of the American Mathematical Society*.

"Why it is Impossible to Trisect an Angle or to Construct a Regular Polygon of Seven or Nine Sides by Ruler and Compasses" will be discussed in the May issue of the Mathematics Teacher by Professor L. E. Dickson, of the University of Chicago.

Some school systems are employing directors or supervisors of mathematics for both their junior and senior high schools.

The Board of Education of Fort Wayne, Indiana, has recently appointed Fred H. Croninger director of mathematics in the city schools.

NOTICES of the following courses in mathematics, to be given during the summer session of 1921, have come to the MATHEMATICS TEACHER:

I. University of Chicago: first term, June 20–July 27; second term, July 28–Sept. 2.

Hermitian matrices of positive type, Professor E. H. Moore.
Determinants, Professor E. H. Moore.
Differential calculus, Professor H. E. Slaught.
Definite integrals, Professor H. E. Slaught.
Seminar on algebra and theory of numbers, Professor L. E. Dickson.
Solid analytic geometry, Professor L. E. Dickson.
Projective differential geometry, Professor E. J. Wilczynski.
College algebra, Professor E. J. Wilczynski.
Applications of vector analysis to electro-magnetism, Professor A. C. Lunn.
Units and dimensions, Professor A. C. Lunn.
Selected topics of mathematics, Professor J. W. A. Young.
Integral calculus, Professor J. W. A. Young.
Selected chapters of algebraic geometry, Professor S. Lefschetz.
Plane analytic geometry, Professor S. Lefschetz.
Functions of a real variable, Professor Henry Blumberg.
Plane trigonometry, Professor Henry Blumberg.

II. Columbia University: July 6–August 12th.

Elementary and intermediate algebra, Professor W. W. Rankin.
Plane geometry, Professor W. W. Rankin.
Logarithms and trigonometry, Professor G. W. Mullins and Drs. J. F. Ritt and K. W. Lamson.
Solid geometry, Dr. Jesse Douglas and Professor W. W. Rankin.
Algebra, Professor W. B. Fite and G. W. Mullins.
Analytical geometry, Professor L. P. Siceloff and Drs. K. W. Lamson and Jesse Douglas.
Calculus, Professor L. P. Siceloff and Dr. G. A. Pfeiffer.
General survey of modern mathematics, Professor Edward Kasner.
Theory of numbers, Dr. J. F. Ritt.

Mathematical introduction to Einstein's theory of relativity, Professor Edward Kasner.

Differential equations, Professor W. B. Fite.

Theory of functions of a real variable, Dr. G. A. Pfeiffer.

The teaching and supervision of arithmetic, Mr. J. R. Clark.

The methods of teaching mathematics in the junior high school, with demonstrations, Mr. Raleigh Schorling.

Theory and practice of teaching algebra in secondary schools, Mr. W. E. Breckenridge.

Demonstration class in plane geometry, Mr. C. B. Walsh.

Practicum in the teaching of mathematics, Professor C. B. Upton.

Commercial mathematics for teachers in high schools and business colleges, Mr. W. S. Schlauch.

A review of the subject-matter of junior high-school mathematics, Mr. C. B. Walsh.

The teaching of applied mathematics, Mr. W. E. Breckenridge.

Industrial mathematics, Mr. W. E. Breckenridge.

The following subjects have been discussed at the Cleveland Mathematics Club during the current year:

Mathematics as a Training for Citizenship,
Correlation of Mathematics with Other Subjects,
Verbal Language Translated into Algebraic Symbols,
Developing the Simple Equation,
Intelligence Tests and Mathematics,
The Aims of Junior High-School Mathematics.
The Aims of Senior High-School Mathematics.
Supervision and Administration of Mathematics,
Factors underlying Failures in Mathematics.

The Cleveland Club is allied with the National Council of Teachers of Mathematics. D. W. Werremeyer, West Technical High School, is president, and Miss Anna T. Campbell, of the Central Junior High School, is secretary. (The average attendance at the monthly meetings has been more than 100.) (By Anna T. Campbell.)

In the February, 1921, issue of *Educational Administration and Supervision* Mr. Percival M. Symonds discussed "Subject Matter Courses in Mathematics for the Professional Preparation of Junior High-School Teachers." This article is of considerable interest, in that the qualifications for teachers of

junior high school mathematics will undoubtedly receive a great deal of attention in the near future. Among other things, Mr. Symonds says:

Our prospective junior high-school teacher of mathematics comes to us then with an equipment of traditional elementary school arithmetic and traditional high-school algebra and geometry. He is to be inspired with a new idea of mathematics, its relation to the rest of life, its place in the school, and the approved methods of teaching it. This new "atmosphere" the normal school must breathe from the start. What higher pure mathematics does the prospective junior high-school teacher need than that which he already has had in the high school? To answer this, ask what concepts and skills and information he needs to have strengthened or even first imparted. We may name (1) formula work, (2) graph work, (3) concept of the function, (4) trigonometry, (5) computation, (6) drawing, (7) properties of solids. For formula work, go to trigonometry; for graph work, go to analytical geometry; for the concept of the function, go to the calculus; for computation, go to logarithms; for drawing and the properties of solids, go to descriptive geometry. Here then is a basis for the work in pure mathematics in the junior high school. Following is the outline of the course suggested:

First year:

Second term: trigonometry and logarithms, with emphasis on the graph, formula, computation, and checking.

Third term: trigonometry, with emphasis on the formula and computation.

Second year:

First term: analytic geometry, with emphasis on the graphs, formula, and concept of the function.

Second term: the calculus, with emphasis on concept of the function, formula and the graph.

Third term: descriptive geometry, with emphasis on drawing and properties of solid bodies.

These courses are not meant to be exhaustive or complete treatments of the subject, but rather selections and adaptations out of the range of material. The work of these five terms is entirely purposive. The teaching has but one point in view: the creation of an appreciative attitude toward the elementary

content of the junior high-school curriculum by thorough grounding in the basic concepts. The instruction should bend every means to this end, and should daily stress the emphasis for which the several courses were planned. The method of attack and means of illustration and development should be models for the future teacher. Wherever possible, teaching principles should be noted.

Besides these advanced courses, however, the student needs a background of the applications of arithmetic. Hence is suggested for the

Second year:

Third term: applications of arithmetic to commercial, economic, and civil problems.

Parenthetically, it may be remarked that this arithmetic course should be social rather than merely arithmetical in content, teaching the facts of everyday life that have an arithmetical bearing.

Third year:

First term: history of mathematics.

Content: A chronological study of the history of mathematics, comparing each area with the present situation, sociologically considered.

Historical development of teaching of arithmetic, algebra, and geometry.

Recent movements and tendencies in the teaching of mathematics.

Third year:

Second and third terms: Curriculum course in junior high-school mathematics.

Content: Reasons for studying and teaching mathematics: the motive and emphasis of mathematics in the junior high school.

The course, responding to present social practice and studying the present course from the historical standpoint.

The approach to different topics, motivation from psychological and historical considerations.

Foreign secondary mathematics, methods of teaching, methods of study, drill, the problem, standards and tests, text books and reference books.

REVIEWS.

Erkenntnislehre im mathematischen Unterricht der Oberklassen. By Dr. W. Lietzmann. Philosophischepädagogische Bibliothek, Band 9, Charlottenburg, 1921. Pp. 68. Price, 12 Marks.

Among those of early middle age who have ranked as the leaders in the teaching of mathematics in Germany in the past fifteen years, no one is better known at home or abroad than Dr. Lietzmann. A disciple of Klein, a successful teacher of mathematics in the secondary schools, an able assistant in the work of the International Commission on the Teaching of Mathematics—he has steadily advanced until he is now director of the Oberrealschule in Göttingen and one of the editors-in-chief of the well-known *Hoffmann's Zeitschrift*, a journal that ranks as one of the best of the world's publications for teachers of secondary mathematics. When such a man writes an article for teachers, they may be well assured that it is worth the reading.

The work sets forth the problem, by no means new, as to the relation of philosophy to education, and particularly to the study of mathematics. What has our science in common with other logical, psychological, and philosophical studies? Indeed, what are the relations of this, a subject of the highest mental type, to other subjects in relatively the same stratum of knowledge? This is quite a different problem from the one that is usually propounded in schools of education, namely, What use will the mass of humanity consciously make of algebra or geometry?

Dr. Lietzmann approaches his problem by the consideration of familiar illustrations taken from the field of elementary mathematics. What, for example, are the basal thought-processes involved in the proof of the Pythagorean Theorem or of a related proposition in solid geometry? What are the logical processes involved in the various proofs of the theorem relating to the sum of the angles of a triangle? What is the

psychological basis upon which mathematical induction rests? And what is the foundation upon which we should build our consideration of the negative number? Such are some of the questions that are raised, inferentially or directly, in the early pages of the work.

The work consists of five chapters, as follows: (1) The proving of propositions and the defining of concepts, in the light of logic; (2) Fundamental propositions and concepts of rational geometry; (3) The number concept; (4) The space of sense perception; (5) The theory of knowledge in education.

Teachers who care to exercise their German in considering the philosophy involved in the presentation of secondary mathematics will find this pamphlet interesting reading. Unfortunately the effects of the war are still manifest in the quality of paper and, to some extent, in the illustrations and the typography.

David Eugene Smith.